

# On brane symmetries

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## Abstract

The geometric approach to branes is reformulated in terms of gauge vector fields interacting with massless tensor multiplets in gravitational backgrounds.

Study of nonlinear dynamics of p-branes [1-14] as well as their quantization require new tools. The geometric approach [15-17] originally developed for strings seems to be relevant for the problem. The gauge reformulation [18] of this approach using the ideas of Cartan [19], Volkov [20] and Faddeev [21] has shown that strings in D-dim. spacetime form a closed sector of states of the exactly integrable two-dimensional  $SO(1, 1) \times SO(D-2)$  gauge model. The geometric approach has turned out to be promising for investigation of integrability of branes PDEs [22-24]. Here we adopt the string gauge approach to p-branes and construct new gauge invariant models which have brane solutions.

1. A time-like  $(p+1)$ -dim. hypersurface  $\Sigma_{p+1}$  embedded into the D-dim. Minkowski spacetime with the signature  $\eta_{mn} = (+, -, \dots, -)$  is described by its radius vector  $\mathbf{x}(\xi^\mu)$  parametrized by the coordinates  $\xi^\mu = (\tau, \sigma^r)$ ,  $(r = 1, 2, \dots, p)$ . Using a local orthonormal frame  $\mathbf{n}_A(\xi^\mu) = (\mathbf{n}_i, \mathbf{n}_a)$  with  $A = (i, a)$ , attached to  $\Sigma_{p+1}$ , one can expand the infinitesimal displacements  $d\mathbf{x}(\xi^\mu)$  and  $d\mathbf{n}_A(\xi^\mu)$  in the local basis  $\mathbf{n}_A(\xi^\mu)$  at the point  $\xi^\mu$

$$d\mathbf{x}(\xi) = \omega^i(\xi)\mathbf{n}_i(\xi), \quad \omega^a(\xi) = 0, \quad (1)$$

$$d\mathbf{n}_A(\xi) = -\omega_A^B(\xi)\mathbf{n}_B(\xi), \quad (2)$$

with the vectors  $\mathbf{n}_i(\xi)$ ,  $(i, k = 0, 1, \dots, p)$  tangent and  $\mathbf{n}_a(\xi)$ ,  $(a, b = p+1, p+2, \dots, D-p-1)$  - normal to the hypersurface. The choice  $\omega^a = 0$  of the normal displacement of  $\mathbf{x}$  breaks down the local Lorentz group  $SO(1, D-1)$  of the moving frame to its subgroup  $SO(1, p) \times SO(D-p-1)$ . Then the antisymmetric matrix differential form  $\omega_{AB} = -\omega_{BA}$  parametrized by  $\xi^\mu$  and belonging to the Lie algebra of  $SO(1, D-1)$  splits into three blocks

$$\omega_A^B \equiv \omega_{\mu A}^B d\xi^\mu = \begin{pmatrix} A_{\mu i}^k & W_{\mu i}^b \\ W_{\mu a}^k & B_{\mu a}^b \end{pmatrix} d\xi^\mu, \quad (3)$$

where  $A_{\mu i}^k$  and  $B_{\mu a}^b$  are transformed as the gauge fields of the  $SO(1, p)$  and  $SO(D-p-1)$  groups on the base space  $\Sigma_{p+1}$ , respectively, and their field strengths  $F_{\mu\nu i}^k$  and  $H_{\mu\nu a}^b$  are

$$F_{\mu\nu i}^k \equiv [D_\mu^\parallel, D_\nu^\parallel]_i^k = (\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]})_i^k, \quad (4)$$

$$H_{\mu\nu a}^b \equiv [D_\mu^\perp, D_\nu^\perp]_a^b = (\partial_{[\mu} B_{\nu]} + B_{[\mu} B_{\nu]})_a^b. \quad (5)$$

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The derivative  $D_\mu^\parallel$  in (4) is covariant with respect to the Lorentz gauge group  $SO(1, p)$  of the subspaces tangent to  $\Sigma_{p+1}$

$$D_\mu^\parallel \phi_\nu^i = \partial_\mu \phi_\nu^i + A_\mu^{\ i}{}_k \phi_\nu^k. \quad (6)$$

The covariant derivative  $D_\mu^\perp$  corresponds to the gauge group  $SO(D - p - 1)$  of rotations of the local subspaces orthogonal to  $\Sigma_{p+1}$

$$D_\mu^\perp \phi_\nu^a = \partial_\mu \phi_\nu^a + B_\mu^{\ a}{}_b \phi_\nu^b. \quad (7)$$

The off-diagonal blocks  $W_{\mu i}^{\ b}$  in (3) are transformed like charged vector multiplets of the gauge group  $SO(1, p) \times SO(D - p - 1)$  with their covariant derivatives

$$(D_\mu W_\nu)_i^{\ a} = \partial_\mu W_{\nu i}^{\ a} + A_{\mu i}^{\ k} W_{\nu k}^{\ a} + B_\mu^{\ a}{}_b W_{\nu i}^{\ b} \quad (8)$$

including the gauge fields  $A_{\mu i}^{\ k}$  and  $B_{\mu a}^{\ b}$ .

The integrability conditions of PDEs (1) and (2) are the Maurer-Cartan (M-C) equations

$$d \wedge \omega_A + \omega_A^{\ B} \wedge \omega_B = 0, \quad (9)$$

$$d \wedge \omega_A^{\ B} + \omega_A^{\ C} \wedge \omega_C^{\ B} = 0 \quad (10)$$

of the structure of the ambient D-dim. space with zero torsion and curvature, where the symbols  $\wedge$  and  $d \wedge$  mean the wedge product and external differential, respectively.

One can see that Eqs. (10), called the Gauss-Codazzi (G-C) equations in the differential geometry of surfaces, contain only the differential form  $\omega_A^{\ B}$ . The splitting (3) of the matrix indices  $A \rightarrow (i, a)$  in (10) results in the field representation of the G-C equations

$$F_{\mu \nu i}^{\ k} = -(W_{[\mu} W_{\nu]})_i^{\ k}, \quad (11)$$

$$H_{\mu \nu a}^{\ b} = -(W_{[\mu} W_{\nu]})_a^{\ b}, \quad (12)$$

$$(D_{[\mu} W_{\nu]})_i^{\ a} = 0, \quad (13)$$

where  $[\mu, \nu]$  means antisymmetrization in  $\mu, \nu$ , e.g.  $\hat{W}_{[\mu} \hat{W}_{\nu]} \equiv \hat{W}_\mu \hat{W}_\nu - \hat{W}_\nu \hat{W}_\mu$ .

For  $p = 1$  the above constraints coincide with the ones discussed upon the gauge reformulation of the geometric approach for strings [18]. This reformulation reveals an isomorphism between the Nambu-Goto string in D-dim. Minkowski space and the exactly solvable sector of the two-dim.  $SO(1, 1) \times SO(D - 2)$  gauge model including a massless scalar multiplet.

Our main goal is to generalize the string case to p-branes which implies construction of a  $(p + 1)$ -dim.  $SO(1, p) \times SO(D - p - 1)$  gauge model compatible with the G-C equations (11-13). This step does not suppose in advance any connections of such a model with the existing models for p-branes, but only takes into account the independence of the constraints (11-13) of the induced metrics of hypersurfaces imbedded into flat spaces. A class of new gauge actions compatible with the G-C constraints is proposed in the next section.

2. The desired  $SO(1, p) \times SO(D - p - 1)$  gauge-invariant action has to describe the gauge and vector fields in an external gravitational field in  $(p + 1)$ -dim. pseudo-Riemannian space with a metric  $g_{\mu\nu}(\xi)$  parametrized by the coordinates  $\xi^\mu$  (that will be later identified with the coordinates parametrizing the brane hypersurface  $\Sigma_{p+1}$ ). The metric  $g_{\mu\nu}$  is not considered

as a dynamical field in contrast to the fields presented in the G-C constraints (11-13). The desired gauge and reparametrization invariant action has the form

$$S = \gamma \int d^{p+1}\xi \sqrt{|g|} \mathcal{L}, \quad (14)$$

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) \\ & + \frac{1}{2} \hat{\nabla}_\mu W_\nu^{ia} \hat{\nabla}^{\{\mu} W_{ia}^{\nu\}} - \hat{\nabla}_\mu W^{\mu ia} \hat{\nabla}_\nu W_{ia}^\nu + V, \end{aligned} \quad (15)$$

where  $\{\mu, \nu\}$  means symmetrization in  $\mu$  and  $\nu$ ,  $V$  encodes nonlinear (self)interactions of the vector multiplet  $W_\mu^{ia}$ . The generalized covariant derivative  $\hat{\nabla}_\mu$  in (15) is

$$\hat{\nabla}_\mu W_{\nu ia} := \partial_\mu W_{\nu ia} - \Gamma_{\mu\nu}^\rho W_{\rho ia} + A_{\mu i}^k W_{\nu ka} + B_{\mu a}^b W_{\nu ib} \quad (16)$$

and extends the general covariant derivative including only the Levi-Chivita connection

$$\nabla_\mu W_{\nu ia} = \partial_\mu W_{\nu ia} - \Gamma_{\mu\nu}^\rho W_{\rho ia}, \quad \nabla_\mu g_{\nu\rho} = 0, \quad (17)$$

where  $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho = \frac{1}{2} g^{\rho\gamma} (\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})$  are the Cristoffel symbols.

The variation of  $S$  (14) in the gauge and vector fields results in the following EOM

$$\hat{\nabla}_\mu F_{ik}^{\mu\nu} = -\hat{\nabla}_\mu (W_{ia}^{[\mu} W^{\nu]a}_{\phantom{a}k}) - \frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[\nu} W^{\mu]a}_{\phantom{a}k]}, \quad (18)$$

$$\hat{\nabla}_\mu H_{ab}^{\mu\nu} = -\hat{\nabla}_\mu (W_{ai}^{[\mu} W^{\nu]i}_{\phantom{i}b}) - \frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}_{\phantom{i}b]} \quad (19)$$

$$\hat{\nabla}_\mu \hat{\nabla}^{\{\mu} W^{\nu\}ia} = 2 \hat{\nabla}^\nu \hat{\nabla}_\mu W^{\mu ia} + \frac{\partial V}{\partial W_{\nu ia}}. \quad (20)$$

With the help of shifted gauge field strengths  $\mathcal{F}_{ik}^{\mu\nu}$  and  $\mathcal{H}_{ab}^{\mu\nu}$

$$\mathcal{F}_{\mu\nu}^{ik} = (F_{\mu\nu} + W_{[\mu} W_{\nu]}^{ik}), \quad (21)$$

$$\mathcal{H}_{\mu\nu}^{ab} = (H_{\mu\nu} + W_{[\mu} W_{\nu]}^{ab}) \quad (22)$$

one can present Eqs. (18-20) in the compact form

$$\hat{\nabla}_\mu \mathcal{F}_{ik}^{\mu\nu} = -\frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[\nu} W^{\mu]a}_{\phantom{a}k]}, \quad (23)$$

$$\hat{\nabla}_\mu \mathcal{H}_{ab}^{\mu\nu} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}_{\phantom{i}b]}, \quad (24)$$

$$\hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} = -2[\hat{\nabla}^\mu, \hat{\nabla}^\nu] W_\mu^{ia} + \frac{\partial V}{\partial W_{\nu ia}}. \quad (25)$$

Further we take into account the generalized first Bianchi identity

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \hat{R}_{\mu\nu} + \hat{F}_{\mu\nu} + \hat{H}_{\mu\nu}, \quad (26)$$

where the Riemann-Cristoffel tensor  $\hat{R}_{\mu\nu} \equiv R_{\mu\nu}{}^\gamma{}_\lambda$  is defined as

$$[\nabla_\mu, \nabla_\nu] V^\gamma = R_{\mu\nu}{}^\gamma{}_\lambda V^\lambda =: (\partial_{[\mu} \Gamma_{\nu]\lambda}^\gamma + \Gamma_{[\mu|\rho}^\gamma \Gamma_{|\nu]\lambda}^\rho) V^\lambda. \quad (27)$$

The identity (26) allows to present Eq. (25) in the form

$$\begin{aligned} & \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} - \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} - \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} \\ &= \frac{1}{2} \frac{\partial V}{\partial W_{\nu ia}} + ([W^\mu, W^\nu], W_\mu)^{ia} - R^{\mu\nu} W_\mu^{ia}, \end{aligned} \quad (28)$$

where  $R_{\nu\lambda} := R^\mu{}_{\nu\mu\lambda}$  is the Ricci tensor. Using the relation

$$\frac{1}{4} \frac{\partial}{\partial W_{\nu ia}} (W_\mu [[W^\mu, W^\rho], W_\rho])^i{}_i = ([W^\mu, W^\nu], W_\mu)^{ia} \quad (29)$$

with the commutators of  $\hat{W}_\mu$  in the r.h.s. we introduce a shifted potential  $\mathcal{V}$

$$\mathcal{V} = V + \frac{1}{2} Sp(W_\mu [[W^\mu, W^\rho], W_\rho]), \quad (30)$$

where the trace  $Sp(W_\mu [[W^\mu, W^\rho], W_\rho]) := (W_\mu [[W^\mu, W^\rho], W_\rho])^i{}_i$ .

As a result, EOM (23), (24) and (28) take the following form

$$\hat{\nabla}_\mu \mathcal{F}^{\mu\nu}{}_{ik} = -\frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[\nu} W^{\mu]a}{}_{|k]}, \quad (31)$$

$$\hat{\nabla}_\mu \mathcal{H}^{\mu\nu}{}_{ab} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}{}_{|b]}, \quad (32)$$

$$\frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} + \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} = \frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu} W_\mu^{ia}. \quad (33)$$

Then we observe that the first-order PDEs which coincide with (11-13)

$$\mathcal{F}^{\mu\nu}{}_{ik} = 0, \quad \mathcal{H}^{\mu\nu}{}_{ab} = 0, \quad \hat{\nabla}^{[\mu} W^{\nu]ia} = 0, \quad (34)$$

form a particular solution of Eqs. (31-33) on condition that

$$\frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu} W_\mu^{ia} = 0. \quad (35)$$

Due to independence of the Ricci tensor  $R^{\mu\nu}$  of  $W_{\nu ia}$ , Eq. (35) allows to restore  $\mathcal{V}$

$$\mathcal{V} = R^{\mu\nu} W_\mu^{ia} W_{\nu ia}. \quad (36)$$

Thus, we find that the action (14) with the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) \\ &+ \frac{1}{2} \hat{\nabla}_\mu W_\nu^{ia} \hat{\nabla}^{\{\mu} W_{ia}^{\nu\}} - \hat{\nabla}_\mu W^{\mu ia} \hat{\nabla}_\nu W_{ia}^\nu \\ &+ R^{\mu\nu} W_\mu^{ia} W_{\nu ia} - \frac{1}{2} Sp(W_\mu [[W^\mu, W^\rho], W_\rho]) \end{aligned} \quad (37)$$

yields the nonlinear Euler-Lagrange equations

$$\hat{\nabla}_\mu \mathcal{F}^{\mu\nu}{}_{ik} = -\frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[\nu} W^{\mu]a}{}_{|k]}, \quad (38)$$

$$\hat{\nabla}_\mu \mathcal{H}^{\mu\nu}{}_{ab} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}{}_{|b]}, \quad (39)$$

$$\frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} + \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} = 0 \quad (40)$$

for the gauge  $A_{\mu i}{}^k$ ,  $B_{\mu a}{}^b$  and vector  $W_{\mu i a}$  fields in a given external gravitational field  $g_{\mu\nu}(\xi^\rho)$ .

It is easy to see that Eqs. (38-40) have the particular solution (34) which coincides with the G-C constraints (11), (12) and (13).

This solves the stated problem of the construction of gauge invariant model compatible with embedded hypersurfaces using the Gauss mapping. In addition note that the action (14) with  $\mathcal{L}$  (37) looks like a natural generalization of the four-dim. Dirac scale-invariant gravity theory with the dynamical dilaton and gravitational field  $g_{\mu\nu}$  (see e.g. [26]).

The above-said hints at consideration of the  $(p+1)$ -dim. spacetime of the gauge model defined by (14), (37) as a  $(p+1)$ -dim. world hypersurface swept by a  $p$ -brane in  $D$ -dim. Minkowski space. Our next step is to prove that the conjecture follows from the remaining Maurer-Cartan eqs. (9) and to find the corresponding modification of the proposed model.

3. To prove the mentioned statement we come back to the M-C Eqs. (9) and split their matrix indices  $A \rightarrow (i, a)$ . This yields the following equations

$$D_{[\mu}^{\parallel} \omega_{\nu]}^i = 0, \quad (41)$$

$$\omega_{[\mu}^i W_{\nu]ia} = 0 \quad (42)$$

with the derivative  $D_{\mu}^{\parallel}$  defined by (6). As shows  $d\mathbf{x}$  squaring, the object  $\omega_{\mu}^i$  plays the role of a  $(p+1)$ -bein for the hypersurface  $\Sigma_{p+1}$  which connects its orthonormal frame  $\mathbf{n}_i$  with the local natural frame  $\mathbf{e}_{\mu}$ , and represents the metric  $G_{\mu\nu}(\xi^\rho)$  of  $\Sigma_{p+1}$  by the quadratic form

$$\omega_{\mu}^i \omega_{\nu}^j = \delta_{\mu\nu}^i, \quad \mathbf{e}_{\mu} = \omega_{\mu}^i \mathbf{n}_i, \quad G_{\mu\nu} = \omega_{\mu}^i \eta_{ik} \omega_{\nu}^k. \quad (43)$$

One can solve the constraints (42) and express  $W_{\mu i}{}^a$  in terms of the symmetric components  $l_{\mu\nu}{}^a$  of the second fundamental form of  $\Sigma_{p+1}$

$$W_{\mu i}{}^a = -l_{\mu\nu}{}^a \omega_i^{\nu}, \quad l_{\mu\nu}{}^a := \mathbf{n}^a \partial_{\mu\nu} \mathbf{x}. \quad (44)$$

The general solution of the constraints (41) is equivalent to the "tetrad postulate"

$$\nabla_{\mu}^{\parallel} \omega_{\nu}^i \equiv \partial_{\mu} \omega_{\nu}^i - \Gamma_{\mu\nu}^{\rho} \omega_{\rho}^i + A_{\mu}{}^i{}_k \omega_{\nu}^k = 0 \quad (45)$$

which identifies the gauge connection  $A_{\mu}{}^i{}_k$  with the background metric connection  $\Gamma_{\mu\nu}^{\rho}$  by means of the gauge transformation

$$\Gamma_{\nu\lambda}^{\rho} = \omega_i^{\rho} A_{\nu}{}^i{}_k \omega_{\lambda}^k + \partial_{\nu} \omega_{\lambda}^k \omega_k^{\rho} \equiv \omega_i^{\rho} D_{\nu}^{\parallel} \omega_{\lambda}^i. \quad (46)$$

Therefore, the hypersurface metric  $G_{\mu\nu}$  has to be identified with the background metric  $g_{\mu\nu}$  introduced *ad hock* in the gauge invariant action (14).

Then the Riemann tensor  $R_{\mu\nu}{}^{\gamma}{}_{\lambda}$  (27) and the field strength  $F_{\mu\nu i}{}^k$  (4) become dependent

$$R_{\mu\nu}{}^{\gamma}{}_{\lambda} = \omega_i^{\gamma} F_{\mu\nu}{}^i{}_k \omega_{\lambda}^k, \quad R_{\nu\lambda} = \omega_i^{\mu} F_{\mu\nu}{}^i{}_k \omega_{\lambda}^k, \quad (47)$$

and the use of the G-C constraint (11) for  $F_{\mu\nu}{}^i{}_k$  allows to express the Ricci tensor as

$$R^{\nu\lambda} = -\omega_{\mu}^i (W^{[\mu} W^{\nu]})_i{}^k \omega_k^{\lambda}. \quad (48)$$

Taking into account (43-47) permits to transit from the gauge  $A_{\nu ik}$  and vector  $W_{\mu i}^a$  fields to the Cristoffel symbols and  $l_{\mu\nu}^a = -\omega_\nu^i W_{\mu i}^a$ , respectively, that transforms (11-13) into

$$R_{\mu\nu}{}^\gamma{}_\lambda = l_{[\mu}{}^\gamma{}_\lambda l_{\nu]}{}^\lambda{}_\lambda, \quad (49)$$

$$H_{\mu\nu}{}^{ab} = l_{[\mu}{}^\gamma{}_\lambda l_{\nu]}{}^\lambda{}_\gamma{}^b, \quad (50)$$

$$\nabla_{[\mu}^\perp l_{\nu]\rho a} = 0, \quad (51)$$

where  $\nabla_\mu^\perp l_{\nu\rho}^a := \partial_\mu l_{\nu\rho}^a - \Gamma_{\mu\nu}^\lambda l_{\lambda\rho}^a - \Gamma_{\mu\rho}^\lambda l_{\nu\lambda}^a + B_\mu^{ab} l_{\nu\rho b}$ .

As is seen, exclusion of  $F_{\mu\nu i}^k$  transforms the constraint (11) into (49) which generalizes the *Gauss Theorema Egregium* for a (p+1)-dim. hypersurface embedded into the D-dim. Minkowski space. The absence of  $F_{\mu\nu i}^k$  allows not to consider the group  $SO(1, p)$  as an explicit symmetry of the desired action. As a result, we obtain the following  $SO(D - p - 1)$  gauge-invariant action in a gravitational background possessing the solution (49-51)

$$\begin{aligned} S &= \gamma \int d^{p+1} \xi \sqrt{|g|} \mathcal{L}, \\ \mathcal{L} &= -\frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) + \frac{1}{2} \nabla_\mu^\perp l_{\nu\rho a} \nabla^\perp \{l^\mu{}^\nu\}^{\rho a} - \nabla_\mu^\perp l_{\rho a}^\mu \nabla_\nu^\perp l^{\nu\rho a} \\ &\quad - \frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b). \end{aligned} \quad (52)$$

To prove this let us consider the following action

$$\begin{aligned} S &= \gamma \int d^{p+1} \xi \sqrt{|g|} \left\{ -\frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) \right. \\ &\quad \left. + \frac{1}{2} \nabla_\mu^\perp l_{\nu\rho a} \nabla^\perp \{l^\mu{}^\nu\}^{\rho a} - \nabla_\mu^\perp l_{\rho a}^\mu \nabla_\nu^\perp l^{\nu\rho a} + V \right\}, \end{aligned} \quad (53)$$

Variation of (53) in the dynamical fields  $l_{\mu\nu}^a$ ,  $B_\mu^{ab}$  gives their EOM

$$\nabla_\nu^\perp \mathcal{H}_{ab}^{\nu\mu} = \frac{1}{2} l_{\nu\rho[a} \nabla^\perp \{l^\nu{}^\mu\}^{\rho b]}, \quad (54)$$

$$\frac{1}{2} \nabla_\mu^\perp \nabla^\perp \{l^\mu{}^\nu\}^{\rho a} = -[\nabla^\perp{}^\mu, \nabla^\perp{}^\nu] l_\mu{}^{\rho a} + \frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}}, \quad (55)$$

where  $\mathcal{H}_{\mu\nu}^{ab} := H_{\mu\nu}^{ab} - l_{[\mu}{}^\gamma{}_\lambda l_{\nu]}{}^\lambda{}_\gamma{}^b$ . Equations (54-55) have the G-C constraints (50-51)

$$\mathcal{H}_{\mu\nu}^{ab} = 0, \quad \nabla_{[\mu}^\perp l_{\nu]\rho a} = 0, \quad (56)$$

as their particular solution provided that

$$\frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}} = [\nabla^\perp{}^\mu, \nabla^\perp{}^\nu] l_\mu{}^{\rho a}. \quad (57)$$

With the help of G-C Eqs. (49-51) and the Bianchi identity

$$[\nabla_\gamma^\perp, \nabla_\nu^\perp] l^{\mu\rho a} = R_{\gamma\nu}{}^\mu{}_\lambda l^{\lambda\rho a} + R_{\gamma\nu}{}^\rho{}_\lambda l^{\mu\lambda a} + H_{\gamma\nu}{}^a{}_b l^{\mu\rho b} \quad (58)$$

one can transform (57) into solvable equation for the self-interaction potential  $V$

$$\frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}} = (l^a l^b)^{\rho\nu} Sp(l_b) + (2l_b l^a l^b - l^a l_b l^b - l_b l^b l^a)^{\rho\nu} - l^{\rho\nu b} Sp(l_b l^a). \quad (59)$$

Equation (59) has the following solution for  $V$  accompanied by the trace constraints

$$V = -\frac{1}{2}Sp(l_a l_b)Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b), \quad Sp(l_a) = 0. \quad (60)$$

The constraints  $Sp(l_a) = 0$  express the well-known algebraic conditions of minimality for a  $(p+1)$ -dim. hypersurface embedded into the Minkowski spaces. These conditions are equivalent to the nonlinear equations of motion of p-branes

$$\square^{(p+1)} \mathbf{x} = 0, \quad (61)$$

where  $\square^{(p+1)} := \frac{1}{\sqrt{|G|}} \partial_\alpha \sqrt{|G|} G^{\alpha\beta} \partial_\beta$  is the reparametrization invariant Laplace-Beltrami operator on  $\Sigma_{p+1}$  [24].

Eq. (61) follows from the Dirac action for p-branes with the minimal hypersurfaces in the Minkowski spacetime

$$S = T \int d^{p+1} \xi \sqrt{|G|}, \quad (62)$$

where  $G$  is the determinant of the induced metric  $G_{\alpha\beta} := \partial_\alpha \mathbf{x} \partial_\beta \mathbf{x}$ .

It proves that the  $SO(D-p-1)$  gauge-invariant action (52) for the interacting gauge and tensor fields  $B_\mu^{ab}$  and  $l_{\mu\nu}^a$ , respectively, in a gravitational background has the particular solution presented by the first-order Gauss-Codazzi PDEs (49-51). The solution describes minimal  $(p+1)$ -dim. hypersurfaces embedded into D-dim. Minkowski spacetime.

**Summary:** The gauge reformulation of the geometric approach to  $(p+1)$ -dimensional hypersurfaces embedded into D-dimensional Minkowski space was proposed. The new set of  $SO(1, p) \times SO(D-p-1)$ -invariant gauge models possessing exact solutions for gauge fields and vector multiplets in gravitational backgrounds, was constructed. The Dirac p-branes were shown to be the solutions of  $(p+1)$ -dimensional gauge model presented by the Gauss-Codazzi constraints for the  $SO(D-p-1)$  gauge vector fields and massless tensor multiplets in curved backgrounds.

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